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LETTER TO THE EDITOR

On the non-integrability of the Störmer problem

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Abstract. In this letter we prove the non-integrability of the Störmer problem, by using the Ziglin–Yoshida method.

The problem of the motion of an electrically charged particle in a magnetic field was first formulated by Störmer [1]. This problem is an important one because of its applications to the case of the Earth's magnetic field. Dragt and Finn [2], by using topological and numerical techniques, showed that, in this case, the motion of trapped charged particles is expected to be non-integrable. Recently Jung and Scholz [3] studied the classical scattering of the same problem and found numerical evidence of chaotic behaviour. See also Jiménez-Lara and Piña [4] for the history of researches on this problem.

The purpose of this letter is to prove the non-integrability of the equations which describe this motion. A non-integrability criterion on the basis of Ziglin's theorem [5] was given by Yoshida [6] for two-degrees-of-freedom Hamiltonians with a homogeneous potential of integer degree. Making the reduction of the original Hamiltonian to a two-dimensional homogeneous problem, we can apply these results and prove the non-integrability of the system. Another proof of non-integrability was given by Noguera [7], who shows the inclusion of the Bernoulli shift as a subsystem of the invariant manifolds of the Lyapunov's orbit in the isolated equilibrium point.

A Hamiltonian system with N degrees of freedom is Liouville integrable whenever N integrals of motion, global, analytical and in involution can be obtained. Furthermore if the level set (intersection) of these integrals is compact and their gradient vectors are linearly independent on the level set for a given initial condition, then the motion is expressed as a quasiperiodic motion on an N-dimensional torus. The proof of the integrability of a Hamiltonian system is based on the exhibition of these N integrals of motion. Several methods for the identification of classes of integrable systems have been developed: Painlevé test [8], symmetry analysis [9], direct construction of invariants [10], etc. However, the proof of the non-integrability is a quite difficult problem in general. In [5] Ziglin has proven a theorem which gives a necessary condition for integrability and can thus be used to prove the non-integrability of a given Hamiltonian system. He considered the monodromy properties around particular solutions and the conditions for the non-existence of an additional integral of motion. In Ziglin's original paper the motion of a rigid body around a fixed point was considered

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as an example. A few applications of Ziglin's approach were given in the last years for Hamiltonian systems with the form: (i) homogeneous potentials [6], (ii) some generalized Toda lattices [11], (iii) some perturbed Kepler potentials [12], (iv) the swinging Atwood's machine [13, 14], (v) non-homogeneous polynomial potentials of degree 3 or 4 [15], and (vi) truncated Toda lattice of any order [16].

Significant advances in the Ziglin's approach were made in the case of homogeneous potentials [6, 17]. In particular, Yoshida proved a theorem concerning the non-integrability of two-dimensional Hamiltonian systems with a homogeneous potential of integer degree, where everything necessary for Ziglin's theorem to be applied is given explicitly.

Yoshida's main theorem [6]. Let $V(q_1, q_2)$ be a homogeneous potential function of an integer degree k and compute the quantity (integrability coefficient) λ defined by

$$\lambda = \mathrm{Tr}[V_{ij}(c_1, c_2)] - (k-1) \tag{1}$$

where V_{ij} is the Hessian matrix of $V(q_1, q_2)$, and (c_1, c_2) is a solution of the algebraic equations

$$\begin{aligned} c_1 &= (\partial V/\partial q_1)(c_1, c_2) \\ c_2 &= (\partial V/\partial q_2)(c_1, c_2). \end{aligned} \tag{2}$$

If λ is in the region S_k defined below, then the two degrees of freedom Hamiltonian system

$$H = (p_1^2 + p_2^2)/2 + V(q_1, q_2) \tag{3}$$

is non-integrable, i.e., there cannot exist an additional integral which is complex analytic in (q_1, q_2) and (p_1, p_2). The regions S_k are defined as follows:

(i) $k \geqslant 3$

$$S_k = \{\lambda < 0,\ 1 < \lambda < k-1, \ldots, j(j-1)k/2 + j < \lambda < j(j+1)k/2 - j, \ldots\};$$

(ii) $S_1 = \boldsymbol{R} - \{0, 1, 3, 6, \ldots, j(j+1)/2, \ldots\};$

(iii) $S_{-1} = \boldsymbol{R} - \{1, 0, -2, -5, \ldots, -j(j+1)/2 + 1, \ldots\};$

(iv) $k \leqslant -3$

$$\begin{aligned} S_k = \{&\lambda > 1,\ 0 > \lambda > -|k| + 2,\ -|k| - 1 > \lambda > -3|k| + 3, \ldots, \\ &-j(j-1)|k|/2 - (j-1) > \lambda > -j(j+1)|k|/2 + (j+1), \ldots\}. \end{aligned} \tag{4}$$

Note that when $k = 0$, -2, $+2$ we have no such regions. Indeed when $k = -2$ every potential is integrable. On the other hand when $k = 0$ and $k = 2$, another analysis is necessary.

Let us consider the non-relativistic motion of a charged particle of mass m and charge q moving in the field of a magnetic dipole of magnetic moment $\boldsymbol{M}$ [2]. It is described by the Hamiltonian

$$H = [\boldsymbol{p} - (q/c)\boldsymbol{A}]^2/2m \tag{5}$$

with

$$\boldsymbol{A} = (\boldsymbol{M} \times \boldsymbol{r})/r^3. \tag{6}$$

By choosing the z axis in the direction of $\boldsymbol{M}$, i.e. $\boldsymbol{M}=(0,0,M)$, we have

$$H=[(p_x+ay/r^3)^2+(p_y-ax/r^3)^2+p_z^2]/2 \tag{7}$$

with $m=1$, $r=[x^2+y^2+z^2]^{1/2}$ and $a=qM/c$.

From (7) we get

$$H=[p_x^2+p_y^2+p_x^2]/2+(a/r^3)(yp_x-xp_y)+(a^2/2r^6)(x^2+y^2). \tag{8}$$

The equations of motion for this system are time-independent, axisymmetric and have also a scale symmetry [18]. The following integrals of motion exist: the Hamiltonian (8) and the projection of the angular momentum in the direction of $\boldsymbol{M}$

$$L_z=xp_y-yp_x. \tag{9}$$

If we choose cylindrical coordinates (ρ, ϕ, z), the Hamiltonian (8) becomes

$$H=[p_\rho^2+p_\phi^2/\rho^2+p_z^2]/2+(a^2\rho^2/2)(\rho^2+z^2)^{-3}+ap_\phi(\rho^2+z^2)^{-3/2} \tag{10}$$

and $p_\phi=L_z=$ constant of motion.

As far as the motion in ρ and z is concerned we can regard $H(p_\rho, p_z, \rho, z, p_\phi=\text{constant})$ as a reduced Hamiltonian describing the two-dimensional motion in the (ρ, z) plane. In particular, if we take $p_\phi=0$ we get the reduced Hamiltonian

$$H=[p_\rho^2+p_z^2]/2+(a^2\rho^2/2)(\rho^2+z^2)^{-3}. \tag{11}$$

If the original Hamiltonian (10) is integrable with a third integral of motion $\Phi=$ constant, then the reduced Hamiltonian (11) should be integrable too. This reduced Hamiltonian (11) has a homogeneous potential with degree $k=-4$:

$$V=(a^2\rho^2/2)(\rho^2+z^2)^{-3}. \tag{12}$$

From (2) we get the algebraic equations

$$\begin{aligned} c_1&=a^2c_1(c_1^2+c_2^2)^{-3}-3a^2c_1^3(c_1^2+c_2^2)^{-4} \\ c_2&=-3a^2c_1^2c_2(c_1^2+c_2^2)^{-4}. \end{aligned} \tag{13}$$

A solution for (13) is

$$c_1=[-2a^2]^{1/6} \text{ and } c_2=0. \tag{14}$$

The Hessian matrix is given by

$$V_{ij}=\begin{bmatrix} -5 & 0 \\ 0 & 3/2 \end{bmatrix} \tag{15}$$

and, therefore, the integrability coefficient λ for the potential (12) will be

$$\lambda=\operatorname{Tr}[V_{ij}(c_1,c_2)]-(k-1)=3/2. \tag{16}$$

In this case, the Yoshida theorem gives us the following non-integrability domains:

$$S_{-4}=\{\lambda>1, 0>\lambda>-2, -5>\lambda>-9, \ldots\}. \tag{17}$$

As λ, from (16), falls in the region S_{-4}, we conclude that the potential (12) is a non-integrable one.

To summarize: we used the Ziglin–Yoshida method for proving the non-integrability of the Störmer problem. Our analysis was performed by using a reduction of the system to a two-dimensional homogeneous potential, at zero angular momentum.

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